Categorical databases

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Purpose of the talk

There is an fundamental connection between databases and categories.

- Category theory can simplify how we think about and use databases.
- We can clearly see all the working parts and how they fit together.
- Powerful theorems can be brought to bear on classical DB problems.

The pros and cons of relational databases

- Relational databases are reliable, scalable, and popular.
- They are provably reliable to the extent that they strictly adhere to the underlying mathematics.
- Make a distinction between
  - the system you know and love, *vs.*
  - the relational model, as a mathematical foundation for this system.

You're not really using the relational model.

- Current implementations have departed from the strict relational formalism:
  - Tables may not be relational (duplicates, e.g from a query).
  - Nulls (and labeled nulls) are commonly used.
- The theory of relations (150 years old) is not adequate to mathematically describe modern DBMS.
- The relational model does not offer guidance for schema mappings and data migration.
- Databases have been intuitively moving toward what's best described with a more modern mathematical foundation.

Category theory gives better description

- Category theory (CT) does a better job of describing what's already being done in DBMS.
  - Puts functional dependencies and foreign keys front and center.
  - Allows non-relational tables (e.g. duplicates in a query).
  - Labeled nulls and semi-structured data fit in neatly.
- All columns of a table are the same type of thing. It's simpler.
- CT offers guidance for schema mapping and data migration.
- It offers the opportunity to deeply integrate programming and data.
- Theorems within category theory, and links to other branches of math (e.g. topology), can be used in databases.

What is category theory?

- Since its invention in the early 1940s, category theory has revolutionized math.
- It's like set theory and logic, except less floppy, more principles-based.
- Category theory has been proposed as a new foundation for mathematics (to replace set theory).
- It was invented to build bridges between disparate branches of math by distilling the essence of mathematical structure.

Branching out

- Category theory naturally fosters connections between disparate fields.
- It has branched out of math and into physics, linguistics, and materials science.
- It has had much success in the theory of programming languages.
- The pure category-theoretic concept of *monads* has vastly extended the reach of functional programming.
- Can category theory improve how we think about databases?

Schemas are categories, categories are schemas

- The connection between databases and categories is simple and strong.
- Reason: categories and database schemas do the same thing.
  - A schema gives a framework for modeling a situation;
    - Tables
    - Attributes
  - This is precisely what a category does.
    - Objects
    - Arrows.
  - They both model how entities within a given context interact.
- Schema = Category.
- In this talk, I'll explain these ideas and some consequences.

Plan of the talk

- Lay out the basic idea of categories and that of databases, and show the tight connection between them.
- Discuss schema evolution and data migration.
- Develop a connection to programming language theory.
- Understand RDF in these terms.

What is a category?

- Idea: A category models entities of a certain sort and the relationships between them.

$$\mathcal{C} := \boxed{\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & C \\ D \arrow[loop left, "i"] \arrow[r, bend left, "j"] & E \arrow[l, bend left, "k"] \end{tikzcd}}$$

- Think of it like a graph: the nodes are entities and the arrows are relationships.
- Some paths can be declared equivalent to others
  - Example: declare that $j; k \simeq i; i; i$ and $f; g \simeq f; h$.

Example

- How could one interpret this kind of abstraction?

$$\mathcal{C} := \boxed{\begin{array}{l} \text{self email is an email from a person} = \\ \text{self email is an email to a person} \\ \begin{tikzcd} \bullet^{\text{Self-email}} \arrow[r, "\text{is an}"] & \bullet^{\text{Email}} \arrow[r, bend left, "\text{from a}"] \arrow[r, bend right, "\text{to a}"'] & \bullet^{\text{Person}} \end{tikzcd} \end{array}}$$

- Such "business rules" can be encoded into the category.

What is the essence of structure?

- If mathematics is the art of getting organized, what organizes math?
- After thousands of years, people realized that there were some essential features in common throughout much of math.
- These are objects, arrows, paths, and path equivalence.
- Or: things, tasks, processes, and "sameness of outcome".
- Or: primary keys, foreign keys, paths of FKs, and path equations.
- Let's give the definition.

Definition of a category I: Constituents

A *category* $\mathcal{C}$ consists of the following constituents:

1. A set $\mathbf{Ob}(\mathcal{C})$, called *the set of objects of* $\mathcal{C}$.
   - (These will be tables.)
   - Objects $x \in \mathbf{Ob}(\mathcal{C})$ is often written as $\bullet^x$.
2. A set $\mathbf{Arr}(\mathcal{C})$, called *the set of arrows of* $\mathcal{C}$, and two functions

   $$src, tgt \colon \mathbf{Arr}(\mathcal{C}) \to \mathbf{Ob}(\mathcal{C}),$$

   assigning to each arrow its *source* and its *target* object, respectively.
   - (Arrows will be foreign keys from "source" table to "target" table.)
   - An arrow $f \in \mathbf{Arr}(\mathcal{C})$ is often written $\bullet^x \xrightarrow{f} \bullet^y$, where $x = src(f), y = tgt(f)$.
   - We define a *path in* $\mathcal{C}$ to be a finite "head-to-tail" sequence of arrows in $\mathcal{C}$, e.g. $\bullet^x \xrightarrow{f} \bullet^y \xrightarrow{g} \bullet^z$.
3. An notion of equivalence for paths, denoted $\simeq$.

Definition of a category II: Rules

These constituents must satisfy the following requirements:

1. If $p \simeq q$ are equivalent paths then the sources agree: $src(p) = src(q)$.
2. If $p \simeq q$ are equivalent paths then the targets agree: $tgt(p) = tgt(q)$.
3. Suppose we have two paths (of any lengths) $b \to c$:

$$\begin{tikzcd}
 & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[dr] & \\
\bullet^{b} \arrow[ur] \arrow[dr] \arrow[rrrr, dashed, bend left, "p"] \arrow[rrrr, dashed, bend right, "q"'] & & & & \bullet^{c} \\
 & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[ur] &
\end{tikzcd}$$

If $p \simeq q$ then for any extensions

$$\begin{tikzcd}
\bullet^{a} \arrow[r, "m"] & \bullet^{b} \arrow[r, dashed, bend left, "p"] \arrow[r, dashed, bend right, "q"'] \arrow[r, phantom, "\simeq"] & \bullet^{c}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet^{b} \arrow[r, dashed, bend left, "p"] \arrow[r, dashed, bend right, "q"'] \arrow[r, phantom, "\simeq"] & \bullet^{c} \arrow[r, "n"] & \bullet^{d}
\end{tikzcd}$$

$m; p \simeq m; q$ and $p; n \simeq q; n$.

What does equivalence of paths mean?

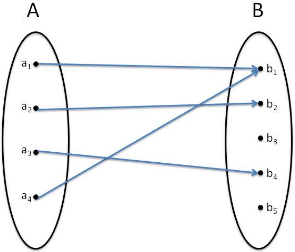
- Arrows represent foreign keys.
- A path $p\colon \bullet^a \to \bullet^b$ represents "following foreign keys" from table a to table b.
- Following a path p, we can take any record in table a and return a record in table b.
- We declare two paths $p, q\colon \bullet^a \to \bullet^b$ equivalent if they should return the same record in b for any record in a.
- In typical DB practices, equivalent paths are avoided by cutting one of the paths.
    - This is considered good design.
    - However, it often causes pain in ones neck.
    - Category theory has this concept built in.

The power of path equivalences

- Ever wanted two directory paths to contain the same file?
- Example: this "Beamer" presentation belongs in my math talks folder and in my J&J consulting folder.
- My file system does not allow that, because without path equivalences, it is dangerous.
- With commutative diagrams we can declare two paths equivalent:

$$
\begin{tikzcd}
\bullet\text{J\&J talks} \arrow[r] \arrow[d] & \bullet\text{J\&J Consulting stuff} \arrow[r] \arrow[d, phantom, "\checkmark"] & \bullet\text{Consulting stuff} \arrow[d] \\
\bullet\text{Math talks} \arrow[r] & \bullet\text{Math stuff} \arrow[r] & \bullet\text{All files (root)}
\end{tikzcd}
$$

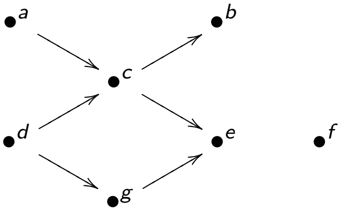
The category of Sets



- Above we see two sets and a function between them. We would denote this categorically by $\bullet^A \xrightarrow{f} \bullet^B$
  - The objects of **Set** represent sets.
  - The arrows in **Set** represent functions.
  - A path represents a sequence of composable functions.
  - Two paths are equivalent if their compositions are the same.
- Note that b_3 and b_5 have been inserted, and a_1 and a_4 have been merged.

A totally different category: an ordered set

- A ordered set is a set S together with a notion of $\leq$, satisfying
  - $a \leq a$ for all $a \in S$, and
  - if $a \leq b$ and $b \leq c$, then $a \leq c$.
- Given some ordered set S, we can build a corresponding category $\mathcal{S}$:
  - $\mathbf{Ob}(\mathcal{S}) = S$,
  - One arrow $a \to b$ if $a \leq b$
  - No arrows $a \to b$ if $a \not\leq b$.
  - All pairs of paths (having same source and target) are equivalent.
- "Hasse diagram":



- Think "permissions": $a \leq c$ means a has fewer accessors than b.

Functors: mappings between categories

- One way to think of a category is as a directed graph, where certain paths have been declared equivalent.
- A functor is a graph mapping that is required to respect equivalence of paths.
- **Definition**: A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of
  - a function $\mathbf{Ob}(\mathcal{C}) \to \mathbf{Ob}(\mathcal{D})$ and
  - a function $\mathbf{Arr}(\mathcal{C}) \to \mathbf{Path}(\mathcal{D})$,

  such that F
  - respects sources and targets,
  - respects equivalences of paths.

Functors to **Set**

- A category $\mathcal{C}$ is a system of objects and arrows, and an equivalence relation on its paths.
- A functor $\mathcal{C} \to \mathcal{D}$ is a mapping that preserves these structures.
- **Set** is the category whose objects are sets, whose arrows are functions, and where paths are equivalent if they compose to the same function.
- If $\mathcal{C}$ is the category on the left below, then a functor $I\colon \mathcal{C} \to \mathbf{Set}$ might look like this:

$$\mathcal{C} := \boxed{\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \end{tikzcd}} \quad ; \qquad I := \boxed{\begin{tikzcd} \boxed{\bullet a_1 \ \bullet a_2 \ \bullet a_3} \arrow[r, "{\begin{smallmatrix} a_1 & \mapsto & b_1 \\ a_2 & \mapsto & b_1 \\ a_3 & \mapsto & b_2 \end{smallmatrix}}"] \arrow[d] & \boxed{\bullet b_1 \ \bullet b_2} \\ \boxed{\bullet c_1} \end{tikzcd}}$$

What is a database?

- A database consists of a bunch of tables and relationships between them.
- The rows of a table are called "records" or "tuples."
- The columns are called "attributes."
- An attribute may be "pure data" or may be a "key."
  - A table may have "foreign key columns" that link it to other tables.
  - A foreign key of table A links into the primary key of table B.
- A schema may have "business rules."

Foreign Keys and business rules

- Example:

Employee				
Id	**First**	**Last**	**Mgr**	**Dpt**
101	David	Hilbert	103	q10
102	Bertrand	Russell	102	x02
103	Alan	Turing	103	q10

Department		
Id	**Name**	**Secr**
q10	Sales	101
x02	Production	102

- Note the Id (primary key) columns and the foreign key columns.
  - Id column could just be internal "row numbers" or could be typed.
  - "Row numbers" (i.e. pointers) are not part of the relational model but they are naturally part of the categorical model.
- Perhaps we should enforce certain integrity constraints (business rules):
  - The manager of an employee E must be in the same department as E,
  - The secretary of a department D must be in D.

Data columns as foreign keys

- Theoretically we can consider a data-type as a 1-column table.
- Examples:

String
a
b
⋮
z
aa
ab
⋮

Integer
0
1
⋮
9
10
11
⋮

- So even data columns can be considered as foreign keys (to respective 1-column tables).
- Conclusion: each column in a table is a key – one primary, the rest foreign.

Example again

Employee				
Id	**First**	**Last**	**Mgr**	**Dpt**
101	David	Hilbert	103	q10
102	Bertrand	Russell	102	x02
103	Alan	Turing	103	q10

Department		
Id	**Name**	**Secr**
q10	Sales	101
x02	Production	102

String
Id
a
b
⋮
z
aa
ab
⋮

$\underline{\text{Mgr;Dpt} \simeq \text{Dpt}}$ $\underline{\text{Secr;Dpt} \simeq \text{id}_{\text{Department}}}$

$$\mathcal{C} := \begin{tikzcd} & \bullet_{\text{Employee}} \arrow[loop above, "\text{Mgr}"] \arrow[dl, "\text{First}"'] \arrow[dr, "\text{Last}"] \arrow[rr, shift left, "\text{Dpt}"] & & \bullet_{\text{Department}} \arrow[ll, shift left, "\text{Secr}"] \arrow[d, "\text{Name}"] \\ \bullet_{\text{String}} & & \bullet_{\text{String}} & \bullet_{\text{String}} \end{tikzcd}$$

Database schema as a category

- A database schema is a system of tables linked by foreign keys.
- This is just a category!

Mgr;Dpt$\simeq$ Dpt   Secr;Dpt$\simeq$ $\text{id}_{\text{Department}}$

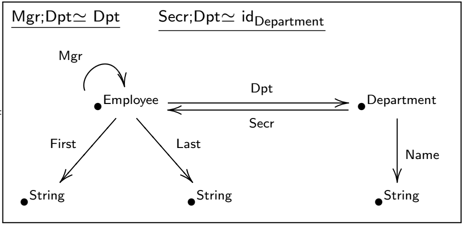
$$\mathcal{C} := \begin{tikzcd} & \bullet\text{Employee} \arrow[loop above, "\text{Mgr}"] \arrow[dl, "\text{First}"'] \arrow[dr, "\text{Last}"] \arrow[rr, shift left, "\text{Dpt}"] & & \bullet\text{Department} \arrow[ll, shift left, "\text{Secr}"] \arrow[d, "\text{Name}"] \\ \bullet\text{String} & & \bullet\text{String} & \bullet\text{String} \end{tikzcd}$$

  - Each object x in $\mathcal{C}$ is a table (Employee, Departments, String);
  - each arrow $x \to y$ is a column of table x.
- Id column of a table corresponds to the trivial path on that object.
- Declaring business rules (e.g. Mgr;Dpt$\simeq$ Dpt) is declaring the path equivalence.

Schema=Category, Instance=Set-valued functor

- Let $\mathcal{C}$ be the following category

$\mathcal{C} :=$



- A functor $I : \mathcal{C} \to \mathbf{Set}$ consists of
  - A set for each object of $\mathcal{C}$ and
  - a function for each arrow of $\mathcal{C}$, such that
  - the declared equations hold.
- In other words, I fills the schema with compatible data.
- Categorical databases could also be called *functional databases*.

Data as a set-valued functor

$\mathcal{C} :=$

$\underline{m; d \simeq d}$ $\underline{s; d \simeq \mathrm{id}_D}$

$$
\begin{tikzcd}
\bullet\text{Emp} \arrow[loop left, "m"] \arrow[rr, shift left, "d"] \arrow[d, "f"'] \arrow[dr, "l"] & & \bullet\text{Dpt} \arrow[ll, shift left, "s"] \arrow[d, "n"] \\
\bullet Str & \bullet Str & \bullet Str
\end{tikzcd}
$$

$I \colon \mathcal{C} \to \mathbf{Set}$

$$
\begin{tikzcd}
\boxed{\text{101,102,103}} \arrow[loop above, "m"] \arrow[rr, shift left, "d"] \arrow[d, "f"'] \arrow[dr, "l"] & & \boxed{\text{q10,x02}} \arrow[ll, shift left, "s"] \arrow[d, "n"] \\
\boxed{\text{a,b,...}} & \boxed{\text{a,b,...}} & \boxed{\text{a,b,...}}
\end{tikzcd}
$$

.

- A category $\mathcal{C}$ is a schema. An object $x \in \mathbf{Ob}(\mathcal{C})$ is a table.
- A functor $I \colon \mathcal{C} \to \mathbf{Set}$ fills the tables with compatible data.
- For each table x, the set $I(x)$ is its set of rows.
- The path equivalences in $\mathcal{C}$ are enforced by I as business rules.

Summary

- The connection between categories and databases is simple.
- A schema is a custom category.
- Functors $I : \mathcal{C} \to \textbf{Set}$ are instances.
- What about functors $F : \mathcal{C} \to \mathcal{D}$ between schemas?

Changes

- We've discussed the situation as though static: a single schema and a single instance.
- Next we'll discuss changes.
- Changing the schema (schema mappings).
- Changing the data (updates).

Changes in schema

- Suppose in our modeling of a given context, we evolve from schema $\mathcal{C}$ to schema $\mathcal{D}$.
- We should find a functorial connection between them.
- Over time we may have something like

$$\mathcal{C} = \mathcal{C}_0 \xrightarrow{F_0} \mathcal{C}_1 \xrightarrow{F_1} \cdots \xrightarrow{F_n} \mathcal{C}_n = \mathcal{D}$$

- We want to be able to migrate data from $\mathcal{C}$ to $\mathcal{D}$ and vice versa.
- We want to be able to migrate queries against $\mathcal{C}$ to queries against $\mathcal{D}$ and vice versa.
- And we want this all to work as it "should".

Composing functors

- Suppose $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$ are functors.
- What is their composition $\mathcal{C} \to \mathcal{E}$?
  - We have a way to take objects in $\mathcal{C}$ to objects in $\mathcal{E}$,
  - Arrows in $\mathcal{C}$ turn into paths in $\mathcal{D}$ and arrows in $\mathcal{D}$ turn into paths in $\mathcal{E}$.
  - We can concatenate these, thus taking arrows in $\mathcal{C}$ to paths in $\mathcal{E}$.
  - Our rules ensure that the equivalences in $\mathcal{C}$ will be preserved in $\mathcal{E}$.
- Composing functors is going to make migrating data more straightforward.

Changes in data

- Let $\mathcal{C}$ be a schema and let $I, J \colon \mathcal{C} \to \mathbf{Set}$ be two instances.
- A *natural transformation* $u \colon I \to J$ consists of the following:
  - For each object (table) $T \in \mathbf{Ob}(\mathcal{C})$ we get a map of record sets
  $$u_T \colon I(T) \to J(T).$$
  - For each arrow (foreign key) $f \colon T \to T'$, we get data consistency; formally,
  $$J(f) \circ u_T = u_{T'} \circ I(f).$$
- If J is the result of an insert or merge (a *progressive update*) to I then
$$u \colon I \to J.$$
- Same thing if I is the result of a delete or a split (a *regressive update*) to J.

The category of instances

- Given a schema $\mathcal{C}$, the *category of instances* on $\mathcal{C}$ is denoted $\mathcal{C}$–**Set**.
  - The objects of $\mathcal{C}$–**Set** are functors (instances) $I\colon \mathcal{C} \to \mathbf{Set}$.
  - The arrows are natural transformations (progressive updates).
  - The paths are sequences of progressive updates.
  - Two paths are equivalent if they result in the same mapping.
- The category $\mathcal{C}$–**Set** is a topos; it has an internal language and logic supporting the *typed lambda calculus*.
- That means, it works well with the theory of programming languages.

Data migration

- Let $\mathcal{C}$ and $\mathcal{D}$ be different schemas.
- We call a functor between them, $F\colon \mathcal{C} \to \mathcal{D}$, a *schema mapping*.
- Given such a mapping, we want to be able to canonically transfer instances on $\mathcal{C}$ to instances on $\mathcal{D}$ and vice versa.
- That means, given $F\colon \mathcal{C} \to \mathcal{D}$ we want functors

$$\mathcal{C}\text{–}\mathbf{Set} \to \mathcal{D}\text{–}\mathbf{Set}$$

and

$$\mathcal{D}\text{–}\mathbf{Set} \to \mathcal{C}\text{–}\mathbf{Set}.$$

What a functor $\mathcal{C}$–**Set** $\to$ $\mathcal{D}$–**Set** means.

A functor $\mathcal{C}$–**Set** $\to$ $\mathcal{D}$–**Set** means:

- **Objects:** To every instance on $\mathcal{C}$ we associate an instance on $\mathcal{D}$.
- **Arrows:** For every progressive update on a $\mathcal{C}$-instance there is a corresponding progressive update on the associated $\mathcal{D}$-instance.
- **Path equivalences:** If two different sequences of progressive updates on $\mathcal{C}$-instances result in the same mapping, then the same will hold of the corresponding sequences on $\mathcal{D}$-instances.

The "easy" migration functor, Δ

- Given a schema mapping (i.e. a functor)

$$F\colon \mathcal{C} \to \mathcal{D},$$

  we can transform instances on $\mathcal{D}$ to instances on $\mathcal{C}$ as follows:

Given $I\colon \mathcal{D} \to \mathbf{Set}$

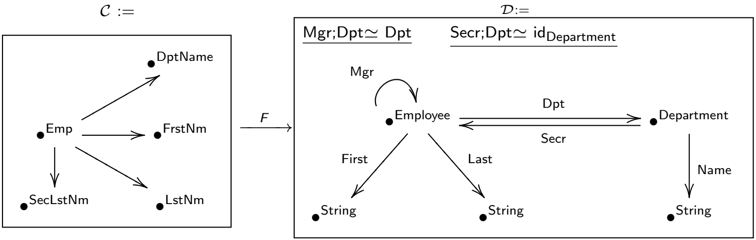
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[rr, bend right, "F;I"'] & \mathcal{D} \arrow[r, "I"] & \mathbf{Set} \end{tikzcd}$$

get $F;I\colon \mathcal{C} \to \mathbf{Set}$

- This process will preserve updates: given an update on I on schema $\mathcal{D}$, it will spit out a corresponding update of $(F;I)$ on schema $\mathcal{C}$.
- Thus we have a functor $\Delta_F\colon \mathcal{D}\text{–}\mathbf{Set} \to \mathcal{C}\text{–}\mathbf{Set}$.

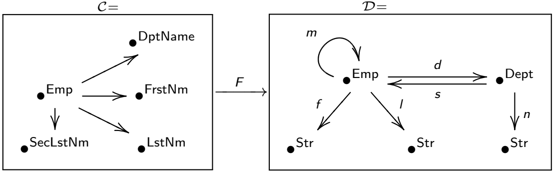
How Δ_F works

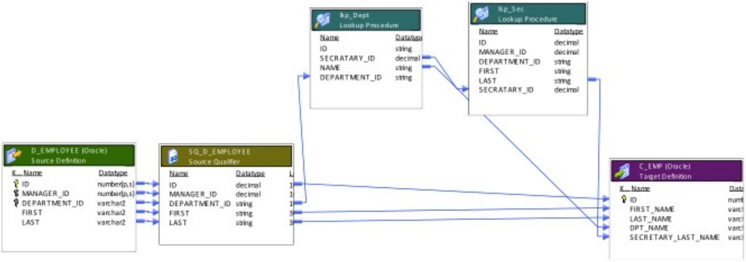
- Consider the schema mapping



- We get $\Delta_F : \mathcal{D}\text{–}\mathbf{Set} \to \mathcal{C}\text{–}\mathbf{Set}$
- Given an instance on $\mathcal{D}$ we get one on $\mathcal{C}$.
- Given an update on $\mathcal{D}$ we get one on $\mathcal{C}$.

Compare the Informatica picture





So many kinds of functors..

- Functors in three different contexts.
  - We started with functors as instances, $I\colon \mathcal{C} \to \textbf{Set}$.
  - Then we introduced functors as schema mappings, $F\colon \mathcal{C} \to \mathcal{D}$.
  - In the last slide we showed a functor on instance categories

$$\Delta_F\colon \mathcal{D}\text{–}\textbf{Set} \to \mathcal{C}\text{–}\textbf{Set}.$$

- Recall the simple definition of functor we gave at the beginning: it holds in each case.
- Functors provide a powerful and reusable abstraction because of the simplicity of their definition.

Adjoints

- Some functors $\mathcal{X} \to \mathcal{Y}$ have a "special partner" $\mathcal{Y} \to \mathcal{X}$ called an *adjoint*.
- What it will mean to us is that we can always "invert" a data migration $\mathcal{D}$–**Set** $\to \mathcal{C}$–**Set** in two universal ways.
  - Roughly, our first inversion will be universal for progressive updates.
  - Our second inversion will be universal for regressive updates.
- These migration functors will provide something like updatable views.
- The important thing is to note is that these aren't made up; they are "canonical" or "universal". They're part of the mathematics – they come with the package.

The "adjoint" migration functors, Σ and Π

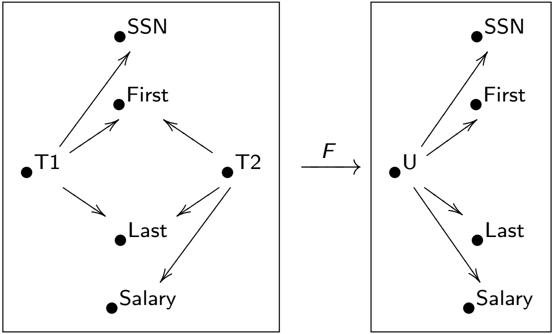
Given a schema mapping (i.e. a functor) $F\colon \mathcal{C} \to \mathcal{D}$,

- We have a functor $\Delta_F\colon \mathcal{D}\text{–}\mathbf{Set} \to \mathcal{C}\text{–}\mathbf{Set}$ given by composition.
- It has two adjoints:
  - a "sum-oriented" adjoint $\Sigma_F\colon \mathcal{C}\text{–}\mathbf{Set} \to \mathcal{D}\text{–}\mathbf{Set}$, and
  - a "product-oriented" adjoint $\Pi_F\colon \mathcal{C}\text{–}\mathbf{Set} \to \mathcal{D}\text{–}\mathbf{Set}$.
- Thus, given a schema mapping F, three functors emerge for the instance categories,

  $$\Delta_F, \Sigma_F, \text{ and } \Pi_F$$

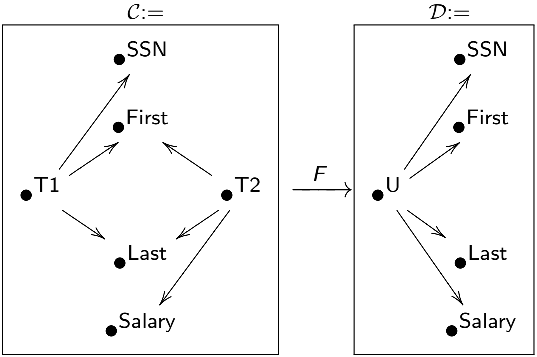
  come with the package.
- Roughly, these correspond to project (Δ), union (Σ), and join (Π).
- They allow one to move data back and forth between $\mathcal{C}$ and $\mathcal{D}$ in canonical ways.

The "product-oriented" push-forward Π_F makes joins



- Given any instance $I : \mathcal{C} \to \mathbf{Set}$, get an instance $\Pi_F(I) : \mathcal{D} \to \mathbf{Set}$.
- The rows in table $\bullet^{\text{U}}$ will be the join of the rows in $\bullet^{\text{T1}}$ and $\bullet^{\text{T2}}$ over $\bullet^{\text{First}}$ and $\bullet^{\text{Last}}$.

The “sum-oriented” push-forward Σ_F makes unions



- Given any instance $I\colon \mathcal{C} \to \textbf{Set}$, get an instance $\Sigma_F(I)\colon \mathcal{D} \to \textbf{Set}$.
- The rows in table $\bullet^{\text{U}}$ will be the union of the rows in $\bullet^{\text{T1}}$ and $\bullet^{\text{T2}}$.
- It will automatically use labeled nulls for the unknown cells.

Views

- These functors can be arbitrarily composed to create views.
- We can think of any series of functors

$$\mathcal{C}_1 \xleftarrow{F_1} \mathcal{D}_1 \xrightarrow{G_1} \mathcal{E}_1 \xrightarrow{H_1} \mathcal{C}_2 \xleftarrow{F_2} \mathcal{D}_2 \xrightarrow{G_2} \cdots \xrightarrow{H_n} \mathcal{C}_n$$

  as a view.
- The view is the functor

$$V := \Sigma_{H_n} \circ \cdots \circ \Pi_{G_1} \circ \Delta_{F_1} : \mathcal{C}_1\text{–}\mathbf{Set} \to \mathcal{C}_n\text{–}\mathbf{Set}.$$

- We can export data from $\mathcal{C}_1$ into $\mathcal{C}_n$ through V.
- Note that $\mathcal{C}_n$ is a schema: not just one table, but possibly many, with foreign keys.
- It's no problem to create views that have foreign keys (unsupported in DBMS).

A simple "SELECT" query using views

SELECT title, isbn
FROM book
WHERE price $>$ 100

$$\mathcal{C} := \begin{tikzcd} & \bullet\text{book} \arrow[r, "\text{title}"] \arrow[d, "\text{price}"] \arrow[dr, "\text{isbn}"] & \bullet\text{String} \\ \bullet\mathbb{R}_{>100} \arrow[r] & \bullet\mathbb{R} & \bullet\text{isbn-num} \end{tikzcd} \xrightarrow{F} \mathcal{D} := \begin{tikzcd} \bullet W \arrow[r] \arrow[d] \arrow[dr, phantom, "\checkmark"] & \bullet\text{book} \arrow[r, "\text{title}"] \arrow[d, "\text{price}"] \arrow[dr, "\text{isbn}"] & \bullet\text{String} \\ \bullet\mathbb{R}_{>100} \arrow[r] & \bullet\mathbb{R} & \bullet\text{isbn-num} \end{tikzcd} \xleftarrow{G} \mathcal{E} := \begin{tikzcd} \bullet W \arrow[r, "\text{title}"] \arrow[dr, "\text{isbn}"] & \bullet\text{String} \\ & \bullet\text{isbn-num} \end{tikzcd}$$

- $V := \Delta_G \circ \Pi_F$ is the appropriate view.
- For any $I \colon \mathcal{C} \to \mathbf{Set}$, we materialize the view as $V(I)$.
- Views with foreign keys are easy.

One more slide about views

- Views can look complex.
  - We can think of any series of functors

$$\mathcal{C}_1 \xleftarrow{F_1} \mathcal{D}_1 \xrightarrow{G_1} \mathcal{E}_1 \xrightarrow{H_1} \mathcal{C}_2 \xleftarrow{F_2} \mathcal{D}_2 \xrightarrow{G_2} \cdots \xrightarrow{H_n} \mathcal{C}_n$$

    as describing a view.
  - In actuality, the view is the functor

$$V := \Sigma_{H_n} \circ \cdots \circ \Pi_{G_1} \circ \Delta_{F_1} : \mathcal{C}_1\text{–}\mathbf{Set} \to \mathcal{C}_n\text{–}\mathbf{Set}.$$

  - We can materialize the view for any $I : \mathcal{C}_1 \to \mathbf{Set}$ as $V(I) : \mathcal{C}_n \to \mathbf{Set}$.
- But a theorem says we can accomplish the same thing in three steps:

$$\mathcal{C}_1 \xleftarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{C}_n$$

- Project – Join – Union.

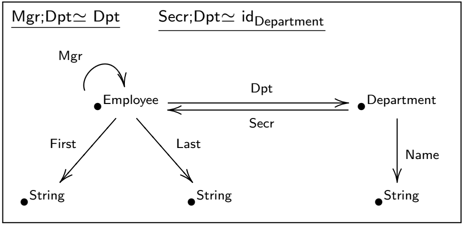
Interfacing between schemas

- We are often interested in taking data from one enterprise model $\mathcal{C}$ and transferring it to another enterprise model $\mathcal{D}$.
- Such transfers can also be accomplished using our notion of views.
- Queries on the old schema translate directly to queries on the new schema.
- We might need to perform calculations such as concatenation, addition, comparison, conversion of units, etc. in order to interface these schemas.
- To do this we'll need an underlying "typing category."

Incorporating data types and functions

- In the example:

$\mathcal{C}=$



how do we know that $\bullet^{\text{String}}$ is what it says it is?

- That is, given $I\colon \mathcal{C} \to \mathbf{Set}$, how do we specify that $I(\bullet^{\text{String}}) \in \mathbf{Set}$ is some pre-defined data type like **String**.

Power of category theory: connection to PL is easy

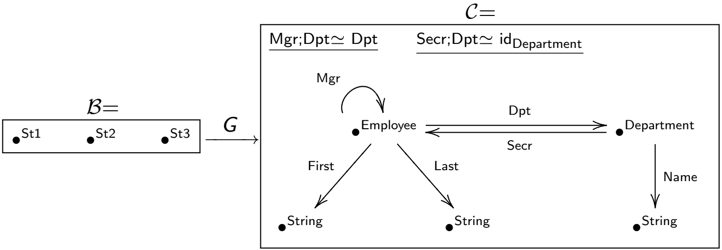
- In programming language theory, they consider the category **Type**.
  - Objects of **Type** are data types, and
  - arrows are functions.
  - Theoretically, there exists a functor $V\colon \mathbf{Type} \to \mathbf{Set}$.
- So **Type** is (in our definition) a database schema and V is a "canonical instance"!
- Since database schemas are categories and **Type** is a category, we can integrate the two.

Example

- Lets make a category $\mathcal{B} = \boxed{\bullet^{\text{St1}} \quad \bullet^{\text{St2}} \quad \bullet^{\text{St3}}}$ and a functor $F \colon \mathcal{B} \to \textbf{Type}$, sending each object to **String** $\in$ **Ob**(**Type**).
- The composition $\mathcal{B} \xrightarrow{F} \textbf{Type} \xrightarrow{V} \textbf{Set}$ yields an instance

$$V' := \Delta_F(V) = V \circ F \colon \mathcal{B} \to \textbf{Set}.$$

- There is also an obvious functor



- A typed instance $I \colon \mathcal{C} \to \textbf{Set}$ is one for which we have a map $\Delta_G(I) \to V'$.

Takeaway

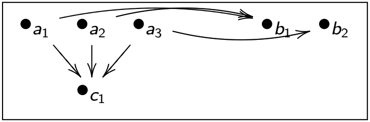
- Databases are custom categories.
- The datatypes in a programming language form a category.
- The whole point of category theory is to allow us to connect different categories.
- Unifying database and program could be very beneficial.

The Grothendieck construction

- Let $\mathcal{C}$ be a category and let $I\colon \mathcal{C} \to \mathbf{Set}$ be a functor.
- We can convert I into a category $Gr(I)$ in a canonical way:
  - Example:

$$\mathcal{C} := \boxed{\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}}\;; \qquad I = \boxed{\begin{tikzcd} \boxed{\bullet a_1 \; \bullet a_2 \; \bullet a_3} \arrow[rr, "{(b_1, b_1, b_2)}"] \arrow[d] & & \boxed{\bullet b_1 \; \bullet b_2} \\ \boxed{\bullet c_1} & & \end{tikzcd}}$$

  - $Gr(I)$ is also known as *the category of elements of* I:



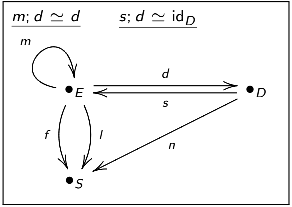
Grothendieck construction applied to database instances

- Suppose given the following instance, considered as $I\colon \mathcal{C} \to \mathbf{Set}$

Employee				
Id	**First**	**Last**	**Mgr**	**Dpt**
101	David	Hilbert	103	q10
102	Bertrand	Russell	102	x02
103	Alan	Turing	103	q10

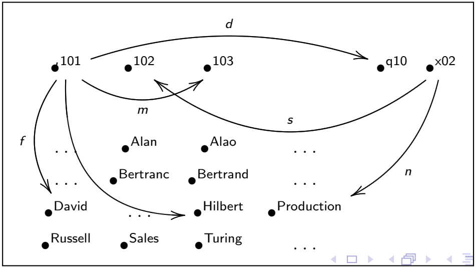
Department		
Id	**Name**	**Secr'y**
q10	Sales	101
x02	Production	102

$\mathcal{C} =$



Here is $Gr(I)$, the category of elements of I:

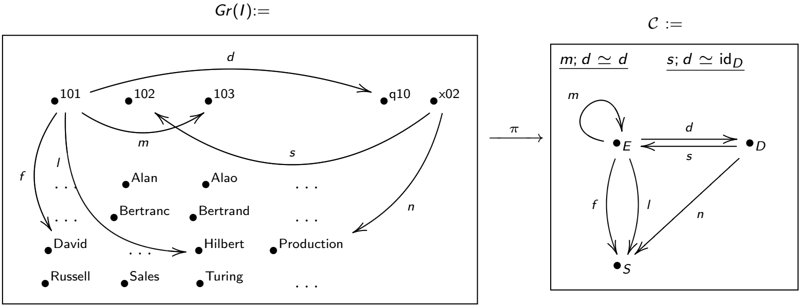
$Gr(I) =$



A different perspective on data

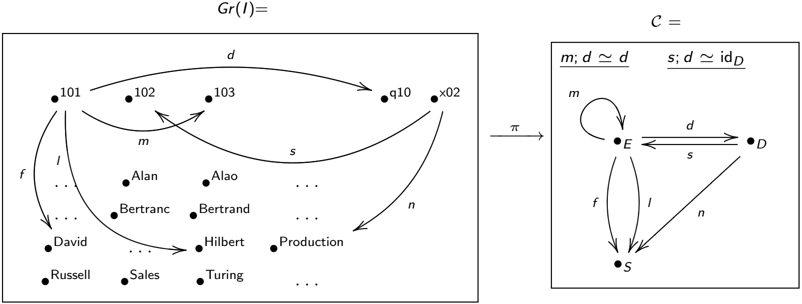
In fact, the Grothendieck construction of $I : \mathcal{C} \to \mathbf{Set}$ always yields not only a category $Gr(I)$ but a functor

$$\pi : Gr(I) \to \mathcal{C}.$$



The fiber over (inverse image of) every object $X \in \mathcal{C}$ is a set of objects $\pi^{-1}(X) \subseteq Gr(I)$. That set is $I(X)$.

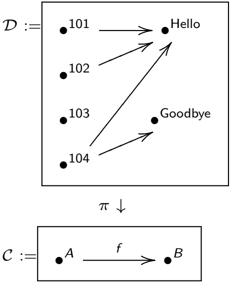
RDF schema and stores



- The relation to RDF triples is clear: each arrow $f: x \to y$ in $Gr(I)$ is a triple with subject x, predicate f, and object y.
- For example (101 department q10), (x02 name Production), etc..
- $\mathcal{C}$ is the RDF schema and $Gr(I)$ is the triple store.
- SPARQL queries (graph patterns) are easily expressible in this model.

Allowing for semi-structured data

- We can think of any functor $\pi\colon \mathcal{D} \to \mathcal{C}$ as a "semi-instance" on $\mathcal{C}$.
- Such a functor π can encode incomplete, non-atomic, or bad data.



- Row 103 has no data in the f cell, and row 104 has too much.
- Bad data (data not conforming to declared path equivalences) can also occur in a functor $\pi\colon \mathcal{D} \to \mathcal{C}$.
- Any semi-instance on $\mathcal{C}$ can be functorially "corrected" to an instance if necessary.
- For example "labeled nulls" will be created for any incomplete data.

Summary

- There's a well-known connection between relational databases and RDF.
- This connection is born out in a most natural way with category theory.
- The model gracefully extends – what should work works.

Summary of the talk

- I hope the connection between databases and categories is clear.

Employee				
Id	**First**	**Last**	**Mgr**	**Dpt**
101	David	Hilbert	103	q10
102	Bertrand	Russell	102	x02
103	Alan	Turing	103	q10

Department		
Id	**Name**	**Secr**
q10	Sales	101
x02	Production	102

$\mathcal{C}=$

$\underline{m; d \simeq d} \qquad \underline{s; d \simeq \mathrm{id}_D}$

$$\begin{tikzcd}
 & E \arrow[loop left, "m"] \arrow[dl, "f"'] \arrow[d, "l"] \arrow[rr, shift left, "d"] & & D \arrow[ll, shift left, "s"] \arrow[d, "n"] \\
S & S & & S
\end{tikzcd}$$

- I discussed how one can use this connection to facilitate:
  - schema mapping and data migration;
  - formalizing views;
  - merging database and programming language theory;
  - merging relational and RDF outlooks;
- The main point is that basic category theory provides a self-contained, unified, and profitable approach to databases.

Thanks for the invitation to speak!