FQL: A Functorial Query Language

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March 4, 2015

1 Syntax and Equational Theory of FQL

The category of finitely presented categories and mappings is bi-cartesian closed, and for every finitely-presented category T, the category of T instances and their morphisms is a topos (bi-cartesian closed category with a subobject-classifier). Hence, FQL has the following structure.

2 Syntax and Equational theory of FQL

Let $\mathcal{T}$ indicate finitely presented categories, $\mathcal{F}_{T_1,T_2} : T_1 \to T_2$ finitely presented functors, $\mathcal{I}_T$ finitely presented T-instances (functors from T to the category of sets), and $\mathcal{E}_{I^1_T, I^2_T} : I^1 \Rightarrow I^2$ finitely presented natural transformations (database homomorphisms from T-instances I^1_T to I^2_T). The syntax of FQL types T, mappings F, instances I, and transformations (database homomorphisms) E is given by the following grammar:

$$T ::= 0 \mid 1 \mid T + T \mid T \times T \mid T^T \mid \mathcal{T}$$

$$F ::= id_T \mid F ; F \mid proj^1_{T,T} \mid proj^2_{T,T} \mid inj^1_{T,T} \mid inj^2_{T,T} \mid F \otimes F \mid F \oplus F \mid ev_{T,T} \mid \Lambda F \mid \mathcal{F}_{T,T} \mid tt_T \mid ff_T$$

$$I ::= 0_T \mid 1_T \mid I + I \mid I \times I \mid I^I \mid \mathcal{I}_T \mid \Omega_T \mid \Delta_F I \mid \Sigma_F I \mid \Pi_F I$$

$$E ::= id_I \mid E ; E \mid proj^1_{I,I} \mid proj^2_{I,I} \mid inj^1_{I,I} \mid inj^2_{I,I} \mid E \otimes E \mid E \oplus E \mid ev_{I,I} \mid \Lambda E \mid \mathcal{E}_{I,I} \mid tt_I \mid ff_I \mid eq_I \mid \top_T$$
$$\mid \Delta_F E \mid \Sigma_F E \mid \Pi_F E \mid \eta^{\Sigma}_{F,I} \mid \epsilon^{\Sigma}_{F,I} \mid \eta^{\Pi}_{F,I} \mid \epsilon^{\Pi}_{F,I}$$

2.1 Isomorphisms of schemas and instances

The following isomorphisms (omit isomorphisms for data migrations Δ, Σ, Π and sub-object classifier Ω) can always be constructed with the appropriate terms:

$$T_1 \times (T_2 \times T_3) \cong (T_1 \times T_2) \times T_3 \qquad T_1 \times T_2 \cong T_2 \times T_1 \qquad T \times 1 \cong 1 \qquad 1^T \cong 1 \qquad T^1 \cong T$$

$$(T_1 \times T_2)^{T_3} \cong T_1^{T_3} \times T_2^{T_3} \qquad (T_1^{T_2})^{T_3} \cong T_1^{T_2 \times T_3} \qquad T_1 + (T_2 + T_3) \cong (T_1 + T_2) + T_3$$

$$T_1 + T_2 \cong T_2 + T_1 \qquad T \times 0 \cong 0 \qquad T + 0 \cong T \qquad T^0 \cong 1 \qquad T_1 \times (T_2 + T_3) \cong (T_1 \times T_2) + (T_1 \times T_3)$$

$$T_1^{T_2 + T_3} \cong T_1^{T_2} \times T_1^{T_3}$$

2.2 Mappings

$$\frac{}{id_T : T \to T} \qquad \frac{F : T_1 \to T_2 \qquad G : T_2 \to T_3}{F ; G : T_1 \to T_3} \qquad \frac{}{tt_T : T \to 1} \qquad \frac{}{proj^1_{T_1,T_2} : T_1 \times T_2 \to T_1}$$

$$\frac{}{proj^2_{T_1,T_2} : T_1 \times T_2 \to T_2} \qquad \frac{F : T_1 \to T_2 \qquad G : T_1 \to T_3}{F \otimes G : T_1 \to T_2 \times T_3} \qquad \frac{}{ff_T : 0 \to T}$$

$$\frac{}{inj^1_{T_1,T_2} : T_1 \to T_1 + T_2} \qquad \frac{}{inj^2_{T_1,T_2} : T_2 \to T_1 + T_2} \qquad \frac{F : T_2 \to T_1 \qquad G : T_3 \to T_1}{F \oplus G : T_2 + T_3 \to T_1}$$

$$\frac{}{ev_{T_1,T_2} : T_1^{T_2} \times T_2 \to T_1} \qquad \frac{F : T_1 \times T_2 \to T_3}{\Lambda F : T_1 \to T_3^{T_2}} \qquad \frac{}{\mathcal{F}_{T_1,T_2} : T_1 \to T_2}$$

$$id; f = f \qquad f; id = f \qquad f;(g;h) = (f;g);h \qquad \Lambda ev = id \qquad \Lambda f \otimes a \ ; \ ev = id \otimes a; f$$

$$f \otimes g; proj^1 = f \qquad f \otimes g; proj^2 = g \qquad f; proj^1 \otimes f; proj^2 = f \qquad \frac{f : T \to 1}{f = tt} \qquad \frac{f : 0 \to T}{f = ff}$$

$$inj^1; f \oplus g = f \qquad inj^2; f \oplus g = g \qquad inj^1; f \oplus inj^2; f = f$$

2.3 Instances

$$\frac{}{0_T : T - inst} \qquad \frac{}{1_T : T - inst} \qquad \frac{I : T - inst \qquad J : T - inst}{I + J : T - inst} \qquad \frac{I : T - inst \qquad J : T - inst}{I \times J : T - inst}$$

$$\frac{I : T - inst \qquad J : T - inst}{I^J : T - inst} \qquad \frac{}{\Omega_T : T - inst} \qquad \frac{}{\mathcal{I}_T : T - inst} \qquad \frac{F : T_1 \to T_2 \qquad I : T_2 - inst}{\Delta_F I : T_1 - inst}$$

$$\frac{F : T_1 \to T_2 \qquad I : T_1 - inst}{\Sigma_F I : T_2 - inst} \qquad \frac{F : T_1 \to T_2 \qquad I : T_1 - inst}{\Pi_F I : T_2 - inst}$$

2.4 Transformations

We omit the equational theory for eq and $\top$, and for the monads (Σ, Δ) and (Δ, Π).

$$\frac{}{id_I : I \Rightarrow I} \qquad \frac{I^1, I^2, I^3 : T - inst \qquad E : I^1 \Rightarrow I^2 \qquad E' : I^2 \Rightarrow I^3}{E; E' : I^1 \Rightarrow I^3} \qquad \frac{I : T - inst}{tt_I : I \Rightarrow 1_T}$$

$$\frac{I^1, I^2 : T - inst}{proj^1_{I^1, I^2} : I^1 \times I^2 \Rightarrow I^1} \qquad \frac{I^1, I^2 : T - inst}{proj^2_{I^1, I^2} : I^1 \times I^2 \Rightarrow I^2}$$

$$\frac{I^1, I^2 : T - inst \qquad E : I^1 \Rightarrow I^2 \qquad E' : I^1 \Rightarrow I^3}{E \otimes E' : I^1 \Rightarrow I^2 \times I^3} \qquad \frac{I : T - inst}{ff_I : 0_T \Rightarrow I} \qquad \frac{I^1, I^2 : T - inst}{inj^1_{I^1, I^2} : I^1 \Rightarrow I^1 + I^2}$$

$$\frac{I^1, I^2 : T - inst}{inj^2_{I^1, I^2} : I^2 \Rightarrow I^1 + I^2} \qquad \frac{I^1, I^2 : T - inst \qquad E : I^2 \Rightarrow I^1 \qquad E' : I^3 \Rightarrow I^1}{E \oplus E' : I^2 + I^3 \Rightarrow I^1}$$

$$\frac{I^1, I^2 : T - inst}{ev_{I^1, I^2} : {I^1}^{I^2} \times I^2 \Rightarrow I^1} \qquad \frac{I^1, I^2, I^3 : T - inst \qquad E : I^1 \times I^2 \Rightarrow I^3}{\Lambda E : I^1 \Rightarrow {I^3}^{I^2}} \qquad \frac{I^1, I^2 : T - inst}{\mathcal{E}_{I^1, I^2} : I^1 \Rightarrow I^2}$$

$$\frac{I : T - inst}{eq_I : I \times I \Rightarrow \Omega_T} \qquad \frac{I : T - inst}{\top_T : 1_T \Rightarrow \Omega_T} \qquad \frac{h : I \Rightarrow J}{\Delta_F h : \Delta_F I \Rightarrow \Delta_F J} \qquad \frac{h : I \Rightarrow J}{\Sigma_F h : \Delta_F I \Rightarrow \Sigma_F J}$$

$$\frac{h : I \Rightarrow J}{\Pi_F h : \Delta_F I \Rightarrow \Pi_F J} \qquad \frac{I : T - inst \qquad F : S \to T}{\eta^{\Sigma}_{F,I} : \Sigma_F \Delta_F I \Rightarrow I} \qquad \frac{I : S - inst \qquad F : S \to T}{\epsilon^{\Sigma}_{F,I} : I \Rightarrow \Delta_F \Sigma_F I}$$

$$\frac{I : S - inst \qquad F : S \to T}{\eta^{\Pi}_{F,I} : \Delta_F \Pi_F I \Rightarrow I} \qquad \frac{I : T - inst \qquad F : S \to T}{\epsilon^{\Pi}_{F,I} : I \Rightarrow \Pi_F \Delta_F I}$$