Categorical Databases

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Outline

An invitation to engage with us, and solve real-world problems.

A road to true interdisciplinarity

- Scientific disciplines are conceptual analogies of the world.
  - Science: a schematic, conceptual account of phenomena.
  - Engineering is using these accounts to channel world events.
  - But how do different disciplines and accounts cohere?
  - To solve big problems, we need to connect different approaches.

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- We need a shared fabric, a substrate for interdiscipinarity.
  - Interdisciplinarity consists of effective analogy-making.
  - To go further, we need to formalize the analogies themselves.
- Better yet: we need a conceptual stem-cell.
  - Something that can differentiate into huge variety of forms.
  - Find the analogies between forms as aspects within the stem cell.

Category theory as conceptual stem-cell

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- All forms of pure math... (we'll briefly discuss this)
- Databases and knowledge representation (categories and functors)
- Functional programming languages (cartesian closed categories)
- Universal algebra (finite-product categories)
- Dynamical systems and fractals (operad-algebras, co-algebras)
- Shannon Entropy (operad of simplices)
- Partially-ordered sets and metric spaces (enriched categories)
- Higher order logic (toposes = categories of sheaves)
- Measurements of diversity in populations (magnitude of categories)
- Collaborative design (enriched categories and profunctors)
- Petri nets and chemical reaction networks (monoidal categories)
- Quantum processes and NLP (compact closed categories)

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  - CT—like math—explains, models, formalizes many many things.
  - Conclude that math/CT explains everything and hence nothing?
- Stem cells don't do work until they differentiate.
  - "Adult-level" work requires differentiation and optimization.
  - But the unified origins lead to impressive interoperability.

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You could also say: CT is mathematics, self-aware.

- Designed to transport theorems from one area of math to another.
  - From topology (shapes) to algebra (equations).
  - This isn't mere analogy, it's analogy made rigorous.

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- And it's branched out from math in a big way.
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If you care about *information hygiene*, CT needs to be on your radar.

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The category-theoretic stem cell is about compositional design patterns.

Let's focus on one: Data frameworks and data transformations.

- The problem: multiple models of similar information
- What is "model-space"?
- Category theory offers a mathematical notion of model-space.
- The kinematics of data: how it moves and rests.

Plan of the talk

- The problem: pervasive and insidious.
- The math: Category theory describes kinematics of data.
- The tool: Open-source implementation and commercialization.

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The Copernican revolution continues

The earth was the center; then the sun; then no need for center.

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- Having a world-center provides an origin; good for coordinating.
- But there isn't just one best coordinate system.
- Each coordinate system is a perspective, a basis for calculation.

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- Linear Algebra studies coordinate systems *and transformations*.
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  - E.g. OMOP in EMRs
  - BFO, CIDOC, SUMO, etc., etc. in upper ontologies
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Multiplicity of perspectives is not going away. Let's learn to integrate.

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  - Finding common structures.
- Information kinematics:
  - Information rests in databases.
  - Information moves by data transformations.
  - Let's dig in.

Information kinematics

Information rests primarily in databases.

- Domain knowledge informs the structure of the database.
    - The structure is called the database *schema*.
    - It consists of a collection of interlocking tables.
- The data itself is structured according to the schema.
    - Unstructured data is not yet informative.
    - It becomes informative through work: structuring it or mining it.

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- A query transforms data from one structure (schema) to another.
  - The result of a query arrives a schema with only one table.
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    - This is a limitation of current notions of querying.
- Other transformations: ETL, schema evolution, warehousing

Think vector spaces and linear transformations.

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- Each table represents some sort of *entity*:
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  - Its columns represents aspects of that entity.
- Example: name and owner are aspects of a house-cat.
  - The house-cat is an entity.
  - The house-cat table has a name column.
  - The house-cat table has an owner column.
  - A house-cat owner is a person, an entity of type person.

House-cat	Name	Owner
C101	Prince Charming	P52
C241	Patches	P52
C468	Mittens	P81

Person	Name
P17	Alice
P52	Bob
P81	Carl

The house-cat schema

Domain knowledge:

- House-cats have names and owners;
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The database collects worldly examples of this knowledge:

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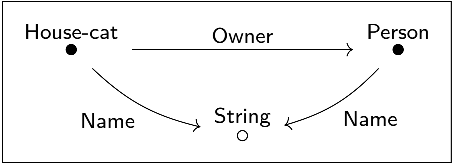
String
Mittens
Patches
⋮

The schema for this knowledge can be drawn as a graph:

$S :=$

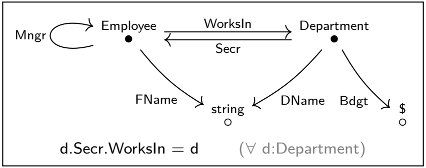


Each column connects its table to another "foreign" table.

A bit more interesting

Let's add loops and integrity constraints:

mySchema :=



Employee	FName	WorksIn	Mngr
1	Alan	101	2
2	Ruth	101	2
3	Kris	102	3

String
Alan
IT
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Department	DName	Secr	Bdgt
101	Sales	1	$10
102	IT	3	$5

x : $
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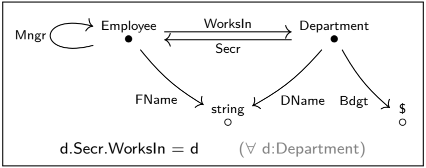

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Stats:

1. Three dots, three tables, three ID columns.
2. Six arrows, six non-ID columns.

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WHERE  e.FName = Alice
RETURN e.WorksIn, e.WorksIn.Secr.FName
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- It's a way of transforming data: form A to form B:

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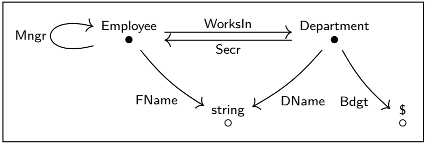
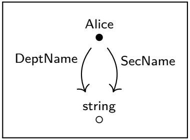


$B :=$



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  - Calculus and diff. eq.? We can hire people to do that.
  - But 40% of IT budgets are spent on information integration.
  - We're constantly breaking and reviving Humpty Dumpty.
  - IT culture has a poor understanding of data transformations.
  - IT culture doesn't even seem to name this problem explicitly.

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  - To properly handle information, we must understand it formally.
  - AI beyond ML requires information agility.

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   assigning to each arrow its *source* and its *target* object, respectively.
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3. An notion of equivalence for paths, denoted $\simeq$.

Definition of a category II: Rules

These constituents must satisfy the following requirements:

1. If $p \simeq q$ are equivalent paths then the sources agree: $src(p) = src(q)$.
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$$\begin{tikzcd}
 & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[dr] & \\
b \arrow[ur] \arrow[dr] \arrow[rrrr, dashed, bend left, "p"] \arrow[rrrr, dashed, bend right, "q"'] & & & & c \\
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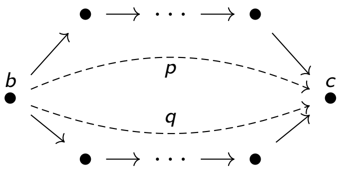
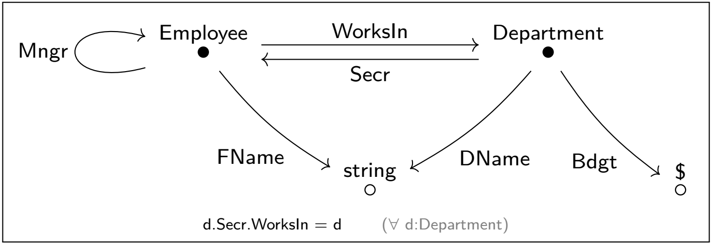


If $p \simeq q$ then for any extensions

$$\begin{tikzcd} a \arrow[r, "m"] & b \arrow[r, bend left, dashed, "p"] \arrow[r, bend right, dashed, "q"'] \arrow[r, phantom, "\simeq"] & c \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} b \arrow[r, bend left, dashed, "p"] \arrow[r, bend right, dashed, "q"'] \arrow[r, phantom, "\simeq"] & c \arrow[r, "n"] & d \end{tikzcd}$$

We have equivalences: $m \fatsemi p \simeq m \fatsemi q$ and $p \fatsemi n \simeq q \fatsemi n$.

Categories = database schemas



- Database schemas are categories!
  - The objects of the category $\mathcal{C}$ are tables.
  - The arrows of $\mathcal{C}$ are columns, connecting one table to another.
  - The integrity constraints are path equations.
  - We brush some details under the rug (distinction between $\bullet$ and $\circ$).

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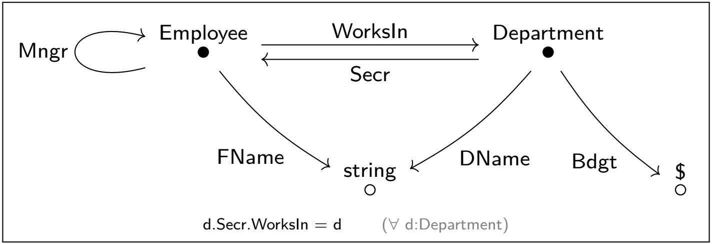

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- But there are also categories that are well-known in math.

Categories from mathematics

Categories are everywhere in mathematics.

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- The category **Set** of sets:
  - Objects $=$ all sets
  - arrows $S \to T =$ all functions from S to T
  - paths $=$ composable functions $S_0 \to S_1 \to \cdots \to S_n$
  - paths equivalent $\iff$ same composite function

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- The category **Vect** of vector spaces
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- The category **Vect** of vector spaces
  - Objects = vector spaces
  - arrows = linear transformations....
- The category of measurable spaces
- The category of metric or topological spaces,
- The category **Cat** of categories.

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  - Objects = all sets
  - arrows $S \to T$ = all functions from S to T
  - paths = composable functions $S_0 \to S_1 \to \cdots \to S_n$
  - paths equivalent $\iff$ same composite function
- The category **Vect** of vector spaces
  - Objects = vector spaces
  - arrows = linear transformations....
- The category of measurable spaces
- The category of metric or topological spaces,
- The category **Cat** of categories.

There's also a notion of *mapping* between categories: functors.

Functors: mappings between categories

What's a functor?

Functors: mappings between categories

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- A functor is a graph mapping that respects path equivalence.

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- Recall: a category is a directed graph with path equivalences.
- A functor is a graph mapping that respects path equivalence.
- **Definition**: A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of
  - a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and
  - a function $\mathrm{Arr}(\mathcal{C}) \to \mathrm{Path}(\mathcal{D})$,

  such that F
  - respects sources and targets,
  - respects equivalences of paths.

Functors and databases

Recall:

- A category $\mathcal{C}$ is basically a database schema.
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- **Set** is a category; recall
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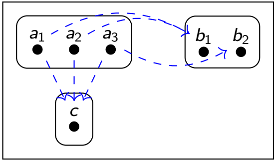
  - its arrows $S \to T$ are functions, and
  - two paths are equivalent if they compose to the same function.
- A functor $\mathcal{C} \to \textbf{Set}$ fills schema $\mathcal{C}$ with data.
  - Example: Let $\mathcal{C}$ be the category on the left.
  - Then here's an example functor $I \colon \mathcal{C} \to \textbf{Set}$:

$$\mathcal{C} := \boxed{\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}}$$

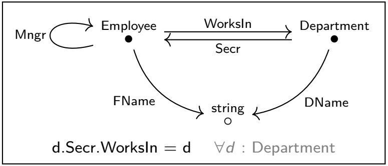
$I :=$



Schema=Category, Instance=Set-valued functor

- Let $\mathcal{C}$ be the following category

$\mathcal{C} :=$



- A functor $I \colon \mathcal{C} \to \mathbf{Set}$ consists of
  - A set for each object of $\mathcal{C}$ and
  - a function for each arrow of $\mathcal{C}$, such that
  - the declared equations hold.
- In other words, I fills the schema with compatible data.

$I :=$

Employee	FName	WorksIn	Mngr
1	Alan	101	2
2	Ruth	101	2
3	Kris	102	3

Department	DName	Secr	Bdgt
101	Sales	1	$10
102	IT	3	$5

Summary of the connection

- The connection between categories and databases is simple.
- A database schema is a custom category.
- Functors $I \colon \mathcal{C} \to \mathbf{Set}$ are database instances.
- What about functors $F \colon \mathcal{C} \to \mathcal{D}$ between schemas?

Data transformations

We want to move data between different frameworks.

- Data is resting in schema $\mathcal{C}$.
- We want to move it in a specific way to schema $\mathcal{D}$.
- We can specify this transformation using functors.

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We can do all sorts of data transformations using functors.

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- But functors play two roles here:
  - They connect schemas to schemas, $F\colon \mathcal{C} \to \mathcal{D}$
  - They connect schemas to data, $I\colon \mathcal{D} \to \mathbf{Set}$.
  - Upshot: one can compose and get a functor $(F \mathbin{⨟} I)\colon \mathcal{C} \to \mathbf{Set}$.

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  - Applying Δ_F to I is called "pulling I back along F".

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  $\Delta_F\colon \mathcal{D}\text{-}\mathbf{Inst} \to \mathcal{C}\text{-}\mathbf{Inst}$.
  - Applying Δ_F to I is called "pulling I back along F".
  - Δ_F has two forward-directional *adjoints*, Σ_F and Π_F.
  - Let's back up a little.

The category of instances

- Given a schema $\mathcal{C}$, the *category of instances* on $\mathcal{C}$ is denoted $\mathcal{C}$-**Inst**.
    - The objects of $\mathcal{C}$-**Inst** are functors (instances) $I\colon \mathcal{C} \to \mathbf{Set}$.
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- Functors between schemas allow us to move data between them.
  - Given a functor $F\colon \mathcal{C} \to \mathcal{D}$,
  - There are automatically three data transformation functors

$$\begin{tikzcd} \mathcal{C}\text{-}\mathbf{Inst} \arrow[r, shift left=3, "\Sigma_F"] \arrow[r, shift right=3, "\Pi_F"'] & \mathcal{D}\text{-}\mathbf{Inst} \arrow[l, "\Delta_F" description] \end{tikzcd}$$

  - Roughly: Δ=project, delete; Σ=sum, union; Π=product, join.
  - Δ, Σ, Π were known by mathematicians in 1960. “Kan extensions”.
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$$\mathcal{C}\text{-}\mathbf{Inst} \;\; \begin{matrix} \xrightarrow{\quad\Sigma_F\quad} \\ \xleftarrow{\quad\Delta_F\quad} \\ \xrightarrow[\quad\Pi_F\quad]{} \end{matrix} \;\; \mathcal{D}\text{-}\mathbf{Inst}$$

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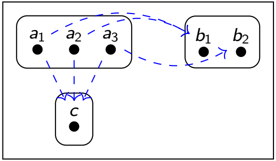
There's a lot of mathematics ready made for hygienically moving data.

Example: The Grothendieck construction

- Let $\mathcal{C}$ be a category and let $I \colon \mathcal{C} \to \textbf{Set}$ be a functor.
- We can convert I into a category $Gr(I)$ in a canonical way:
    - Example:

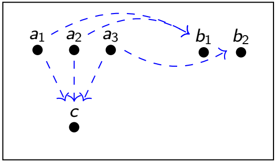
$$\mathcal{C} := \boxed{\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}}$$

$I :=$



- $Gr(I)$ is also known as *the category of elements of I*:

$Gr(I) =$



This applies to database instances

Suppose given the following instance, considered as $I: \mathcal{C} \to \mathbf{Set}$

Employee			
Id	**FName**	**Mgr**	**WorksIn**
1	Alan	3	101
2	Ruth	2	102
3	Kris	3	101

Department		
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101	Sales	1
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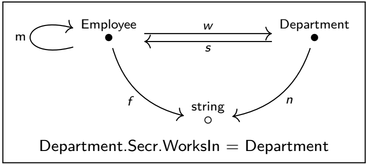
$\mathcal{C} :=$



Here is $Gr(I)$, the category of elements of I:

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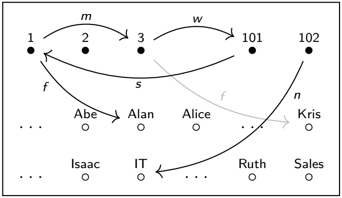
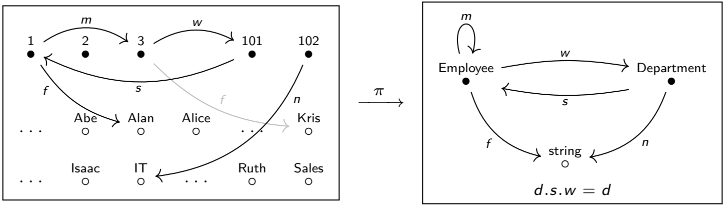


Relations to RDF

The category of elements comes with a functor $\pi\colon Gr(I) \longrightarrow \mathcal{C}$.



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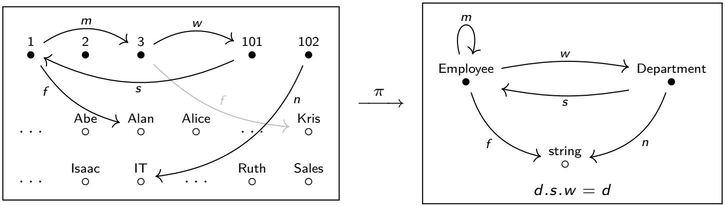

Relation to RDF triple stores and schemas:

- Each arrow $x \xrightarrow{f} y$ in $Gr(I)$ is an *RDF triple* (x, f, y).
  - subject$=x$, predicate$=f$, object $=y$.
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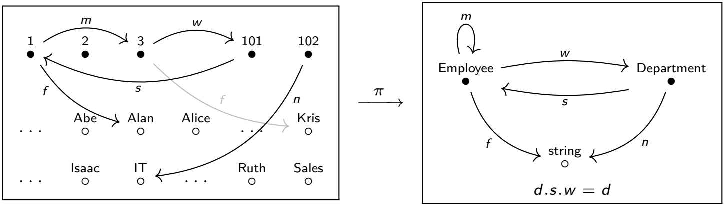

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- Category theoretic model of RDF
  - Think of $Gr(I)$ as RDF triple store, $\mathcal{C}$ as RDF schema.
  - SPARQL graph pattern queries fit easily into the model.
  - Models embedded dependencies (analogous to OWL schemas).

Outline

An invitation to engage with us, and solve real-world problems.

The history of AQL

- The mathematical foundations of this story are old.
  - The basic idea was known to mathematicians 60 years ago.
  - More recently we've learned a lot about how to calculate them fast.
  - Getting data types ($\circ^{\text{string}}$) into the picture is a little more delicate.

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- We received funding from various government agencies.
  - ONR, AFOSR, NIST, NSF.
- A company spun out of MIT in 2015.
  - Categorical Informatics Inc.
  - All MIT IP is open source, all Catinf IP is not.

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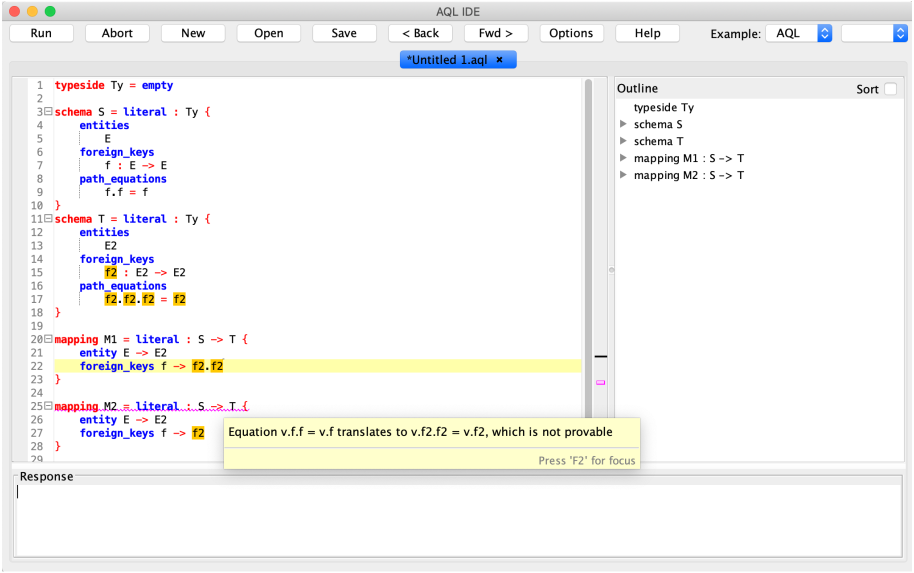
- And more (natural transformations, algebraic theories, profunctors, Grothendieck construction, (co-) monads..., simply-typed lambda calculus)

Screenshot


AQL IDE
Run
Abort
New
Open
Save
< Back
Fwd >
Options
Help
Example: AQL
*Untitled 1.aql
typeside Ty = empty
schema S = literal : Ty {
entities
E
foreign_keys
f : E -> E
path_equations
f.f = f
}
schema T = literal : Ty {
entities
E2
foreign_keys
f2 : E2 -> E2
path_equations
f2.f2.f2 = f2
}
mapping M1 = literal : S -> T {
entity E -> E2
foreign_keys f -> f2.f2
}
mapping M2 = literal : S -> T {
entity E -> E2
foreign_keys f -> f2
}
Equation v.f.f = v.f translates to v.f2.f2 = v.f2, which is not provable
Press 'F2' for focus
Outline
Sort
typeside Ty
schema S
schema T
mapping M1 : S -> T
mapping M2 : S -> T
Response

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- CT been recently highlighted by agencies such as NIST and DARPA.
  - CT stem cell leads to interoperability and compositionality.
  - It compresses and connects big ideas.
  - It helps you take care of all the corner cases.
  - Through strong abstraction principles, it exposes conceptual neighbors.

Summary of the talk

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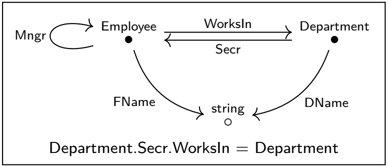

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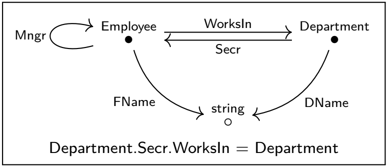

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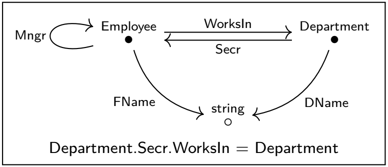

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- Information kinematics—how data moves—is well-modeled by CT.
- With a good understanding, we save a lot of time and effort.

For more...

Book: *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press, July 2019. https://arxiv.org/abs/1803.05316

Company: Categorical Informatics. Website: http://catinf.com

Community: Category Theory Seminar, Thursdays 4:30 – 5:30, MIT Building 2, room 255. http://brendanfong.com/seminar.html

Thanks for the invitation to speak!